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A class of strongly stable approximation for unbounded operators

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Класс сильно устойчивой аппроксимации неограниченных операторов

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Abstract. We derive new sufficient conditions to solve the spectral pollution problem by using the generalized spectrum method. This problem arises in the spectral approximation when the approximate matrix may possess eigenvalues which are unrelated to any spectral properties of the original unbounded operator. We develop the theoretical background of the generalized spectrum method as well as illustrate its effectiveness with the spectral pollution. As a numerical application, we will treat the Schrödinger's operator where the discretization process based upon the Kantorovich's projection.

Keywords: eigenvalue approximation; spectral pollution; generalized spectrum approximation, Schrödinger operator

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Аннотация. С использованием метода обобщенного спектра получены новые достаточные условия решения проблемы спектрального загрязнения. Эта проблема, возникающая в спектральном приближении, вызвана тем, что приближенная матрица может иметь собственные значения, которые не связаны с какими-либо спектральными свойствами исходного неограниченного оператора. Мы разрабатываем теоретические основы метода обобщенного спектра, а также иллюстрируем его эффективность при наличии спектрального загрязнения. В качестве численного приложения рассматривается оператор Шрёдингера, а процесс дискретизации этого оператора основывается на проекции Канторовича.

Ключевые слова: приближение собственных значений; спектральное загрязнение; аппроксимация обобщенного спектра; оператор Шрёдингера

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1. Introduction

Let $(H, D(H))$ be a self-adjoint unbounded operator on a Hilbert space $\mathcal{H}$. With the purpose of finding the spectrum set $sp(H)$ of the operator H by using numerical approach, the conventional methods used are the projection methods (see e.g. [1] and [2]). Precisely, let $(P_k)_{k\in\mathbb{N}}$ be a sequence of orthogonal projections $P_k : \mathcal{H} \to \mathcal{L}_k$, where the closed set $\mathcal{L}_k$ is a subspace of $D(H)$. In the theory of spectral approximation, we seek whether or not $\lim\limits_{k\to\infty} sp(P_k\, H\, P_k) = sp(H)$. Generally, the result is negative, where for k large enough, the set $sp(H_k)$ may contain points that do not belong to the set $sp(H)$.

The weakness of projection method is well known in numerical analysis as the spectral pollution problem, this is an important problem in several areas in the field of applied mathematics (see e.g. [3], [4] and [5]).

In this paper, we use an alternative method, the *generalized spectral method*, which has been introduced in [6]. This new method is based on the concept of the *generalized spectrum* (see [7] and [8]).

Let T and S be two bounded linear operators defined on a Banach space $\mathcal{X}$, we define the *generalized resolvent*,

$$re(T, S) = \{z \in \mathbb{C} : (T - zS) : \mathcal{X} \to \mathcal{X} \text{ is beijective } \}.$$

The complementary set of the *generalized resolvent* set is the *generalized spectrum*, denoted $sp(T, S)$. We say that λ is a *generalized eigenvalue* of the couple (T, S) if there exists $u \in \mathcal{X} \setminus \{0\}$ such that

$$Tu = \lambda Su.$$

The subspace $\mathrm{Ker}(T-\lambda S)$ is called the *generalized spectral subspace* corresponding to λ.

The space of all bounded linear operator defined on the Banach space $\mathcal{X}$ is denoted by $BL(\mathcal{X})$. We consider now an unbounded operator $(A, D(A))$ defined on $\mathcal{X}$, we recall that the resolvent set of A is given by

$$re(A) = \{z \in \mathbb{C} : (A - zI) : D(A) \to \mathcal{X}, \text{ is beijective and } (A - zI)^{-1} \in BL(\mathcal{X})\},$$

and the spectrum set of A is $sp(A) = \mathbb{C} \setminus re(A)$.

In this work, under the assumption $re(A) \neq \emptyset$, we prove that each spectral problem associated to A has an equivalent *generalized spectral problem* which means that there exist two bounded operators T and S defined on $\mathcal{X}$, satisfying $sp(T,S) = sp(A)$. Furthermore, if λ is an eigenvalue of A, then λ is a *generalized eigenvalue* of the couple (T,S) and

$$\mathrm{Ker}(A - \lambda I) = \mathrm{Ker}(T - \lambda S). \tag{1.1}$$

Through the numerical approximation of the bounded operators T and S by sequences of bounded operators $(T_k)_{k\in\mathbb{N}}$ and $(S_k)_{k\in\mathbb{N}}$ defined on $\mathcal{X}$, where they converge in an appropriate sense to T and S, we prove that

$$\lim_{k\to\infty} sp(T_k, S_k) = sp(T,S).$$

The limit here is understood as a combination of the following Property U and Property L, where they are naturally extended from the classical case with $S = I$ (see [9]).

Property U: if $\lambda_k \in sp(T_k, S_k)$ and $\lambda_k \to \lambda$, then $\lambda \in sp(T,S)$.

Property L: if $\lambda \in sp(T,S)$, then there exists $(\lambda_k)_{k\in\mathbb{N}}$ such that $\lambda_k \in sp(T_k, S_k)$ and $\lambda_k \to \lambda$.

We organize this paper as follows: throughout section 2, we construct the theoretical foundations of the generalized spectral method. This theory is a generalization of the classical case when $S = I$ (see [9]). In section 3, we prove that the Property U and Property L hold under appropriate convergence of $(T_k)_{k\in\mathbb{N}}$ and $(S_k)_{k\in\mathbb{N}}$ to T and S respectively. Finally, a numerical application is given for the case of Schrödinger's operator, where our numerical results show the coherence and the effectiveness of the generalized spectrum method (see [11]).

2. Generalized spectrum

Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space. The space $BL(\mathcal{X})$ is the set of all bounded linear operators on $\mathcal{X}$ equipped with the subordinated operator norm,

$$\|A\| = \sup\{\|Ax\| : x \in \mathcal{X},\ \|x\| = 1\},\ A \in BL(\mathcal{X}).$$

Let T and S be two operators in $BL(\mathcal{X})$, for $z \in re(T,S)$, we set

$$R(z,T,S) = (T - zS)^{-1}$$

as *the generalized resolvent operator*. Let $\lambda \in sp(T,S)$ be a generalized eigenvalue, we say that λ has a *finite algebraic multiplicity* if

$$\dim \operatorname{Ker}(T-\lambda S) < \infty.$$

We remark that, if the operator S is invertible, then

$$sp(T,S) = sp(S^{-1}T),$$

but if S^{-1} does not exist, the generalized spectrum set can be a bounded set, or the whole $\mathbb{C}$, or an empty set.

The next three results are a generalization of a classical case when $S = I$. The proofs are provided in [6].

Theorem 2.1. *Let $\lambda \in re(T,S)$ and $\mu \in \mathbb{C}$, where $|\lambda - \mu| < ||R(\lambda,T,S)S||^{-1}$. Then $\mu \in re(T,S)$.*

Corollary 2.1. *The set $sp(T,S)$ is closed in $\mathbb{C}$.*

Theorem 2.2. *The function $R(\cdot,T,S) : re(T,S) \to BL(\mathcal{X})$ is analytic, and its derivative is given by $R(\cdot,T,S)SR(\cdot,T,S)$.*

We consider now an unbounded operator A with domain $D(A) \subset \mathcal{X}$. The following theorem shows that every unbounded operator allows a pair of two bounded operators in $BL(\mathcal{X})$ which expresses it in the terms of the generalized spectrum.

Theorem 2.3. *If $re(A) \neq \emptyset$, then there exist $T,S \in BL(\mathcal{X})$ such that*

$$sp(A) = sp(T,S).$$

In particulary, λ is an eigenvalue for A if and only if λ is a generalized eigenvalue for the couple (T,S). In addition, the equality (1.1) is satisfied.

P r o o f. Let $\alpha \in re(A)$. We define $S,T : \mathcal{X} \to D(A)$ as follows:

$$S = (A-\alpha I)^{-1},\, T = A\,(A-\alpha I)^{-1}.$$

It is clear that $T,S \in BL(\mathcal{X})$. To show that $sp(A) = sp(T,S)$, we prove that $re(A) = re(T,S)$. Let $\lambda \in re(A)$, i. e. there exists operator $(A-\lambda I)^{-1} \in BL(\mathcal{X})$. Then

$$(A-\lambda I)(A-\alpha I)^{-1} = I + (\alpha-\lambda)(A-\alpha I)^{-1} \in BL(\mathcal{X}).$$

So as

$$[(A-\lambda I)(A-\alpha I)^{-1}]^{-1} = (A-\alpha I)(A-\lambda I)^{-1} = I + (\lambda-\alpha)(A-\lambda I)^{-1} \in BL(\mathcal{X}),$$

we get

$$A(A-\alpha I)^{-1}-\lambda(A-\alpha I)^{-1}\in BL(\mathcal{X})\Rightarrow(T-\lambda S)^{-1}\in BL(\mathcal{X}).$$

Thus, it is proved that $\lambda\in re(T,S)$.

Inversely, let $\lambda\in re(T,S)$. To show that $(A-\lambda I)^{-1}\in BL(\mathcal{X})$, we prove that $(A-\lambda I)$ is bijective. Firstly, check the injectivity. Let $u\in D(A)$, using the fact that A commutes with $(A-\alpha I)^{-1}$ we have

$$(A-\alpha I)^{-1}Au=A(A-\alpha I)^{-1}u=u+\alpha(A-\alpha I)^{-1}u. \tag{2.2}$$

Taking into consideration the equality (2.2), we find

$$\begin{aligned}(A-\lambda I)u=0 &\Rightarrow (A-\alpha I)^{-1}(A-\lambda I)u=0\Rightarrow(A-\lambda I)(A-\alpha I)^{-1}u=0\\ &\Rightarrow [A(A-\alpha I)^{-1}-\lambda(A-\alpha I)^{-1}]u=0\Rightarrow(T-\lambda S)u=0\Rightarrow u=0.\end{aligned}$$

Secondly, prove the surjectivity. For all $y\in\mathcal{X}$ we show that $(A-\lambda I)x=y$ has a solution $x\in D(A)$. Put $x=(A-\alpha I)^{-1}(T-\lambda S)^{-1}y$; it is clear that $x\in D(A)$ (the fact that $(A-\alpha I)^{-1}:\mathcal{X}\to D(A)$), moreover we have

$$\begin{aligned}(A-\lambda I)(A-\alpha I)^{-1}(T-\lambda S)^{-1}y &= [A(A-\alpha I)^{-1}-\lambda(A-\alpha I)^{-1}](T-\lambda S)^{-1}y\\ &= (T-\lambda S)(T-\lambda S)^{-1}y=y.\end{aligned}$$

Furthermore, we can see, upon the choice of the vector x, that

$$\|x\|\leq\|(A-\alpha I)^{-1}\|\ \|(T-\lambda S)^{-1}\|\ \ \|y\|,$$

so

$$\|(A-\lambda I)^{-1}\|\leq\|(A-\alpha I)^{-1}\|\ \|(T-\lambda S)^{-1}\|,$$

which implies that $(A-\lambda I)^{-1}\in BL(\mathcal{X})$ and therefore $\lambda\in re(A)$.

Now, we show that the equality (1.1) holds. Let λ be a generalized eigenvalue of the couple (T,S), then there exists $u\in\mathcal{X}\backslash\{0\}$ such that $Tu=\lambda Su$, thus

$$\begin{aligned}Tu=\lambda Su &\Rightarrow A(A-\alpha I)^{-1}u=\lambda(A-\alpha I)^{-1}u\\ &\Rightarrow u=(\lambda-\alpha)(A-\alpha I)^{-1}\Rightarrow u\in D(A).\end{aligned}$$

By applying $(A-\alpha I)$ on $Tu=\lambda Su$, we find that $Au=\lambda u$. Inversely, let λ be an eigenvalue of A, then $Au=\lambda u$. So, by applying $(A-\alpha I)^{-1}$ on $Au=\lambda u$ and using the fact that $(A-\alpha I)^{-1}Au=A(A-\alpha I)^{-1}u$ for all $u\in D(A)$, we find that $Tu-\lambda Su=0$. □

We note that the choice of the couple (T,S) as a function of the resolvent operator of A is not unique (see the numerical application below).

The next results represent the theoretical background of the generalized spectrum approach.

Theorem 2.4. *Let $T, S \in BL(\mathcal{X})$, and let λ be a generalized eigenvalue with finite algebraic multiplicity, isolated in $sp(T,S)$. We denote by Γ the Cauchy contour separating λ from $sp(T,S)$. Then the operator*

$$P = \frac{-1}{2i\pi}\int_\Gamma (T - zS)^{-1} S\, dz \tag{2.3}$$

defines a projection from $\mathcal{X}$ to $\mathcal{X}$, and we have

$$P\mathcal{X} = Ker(T - \lambda S). \tag{2.4}$$

P r o o f. To show that the operator P given by (2.3) is a projection form $\mathcal{X}$ to $\mathcal{X}$, see the book [8, p. 50]. Now to prove the equality (2.4), firstly we fix $\alpha \in re(T,S)$, where for any Cauchy contour Γ associated with λ we assume that $\alpha \notin \Gamma$. For $\mu \in \Gamma$, we have

$$\mu S - T = (\alpha S - T)[(\alpha-\mu)^{-1} I - (\alpha S - T)^{-1} S](\alpha - \mu)$$

which gives

$$(\mu S - T)^{-1} = [(\alpha-\mu)^{-1} I - (\alpha S - T)^{-1} S]^{-1}(\alpha-\mu)^{-1}(\alpha S - T)^{-1}.$$

Thus, we can see that $(\alpha-\lambda)^{-1}$ is an eigenvalue of the operator $(\alpha S - T)^{-1} S$. Indeed

$$\begin{aligned} u \in \mathrm{Ker}(T-\lambda S) \;&\Rightarrow\; (T-\lambda S)u = 0 \Rightarrow (\alpha S - T)^{-1}(\alpha S - T + T - \lambda S)u = u \\ &\Rightarrow\; (\alpha S - T)^{-1} S u = (\alpha-\lambda)^{-1} u \Rightarrow u \in \mathrm{Ker}((\alpha S - T)^{-1} S - (\alpha-\lambda)^{-1} I). \end{aligned}$$

We reverse the last process and get

$$\mathrm{Ker}(T - \lambda S) = \mathrm{Ker}((\alpha S - T)^{-1} S - (\alpha-\lambda)^{-1} I).$$

Now, under the choice of α, we can see that for all Cauchy contour Γ, $\eta(\Gamma)$ is also a Cauchy contour of the eigenvalue $(\alpha-\lambda)^{-1}$ where $\eta(\mu) = (\alpha-\mu)^{-1}$.

We put $B = (\alpha S - T)^{-1} S$ and $z = (\alpha-\mu)^{-1}$ for any $\mu \in \Gamma$. Following this notation we have

$$(\mu S - T)^{-1} S = z[-I + z(zI - B)^{-1}].$$

Thus, integrating over the Γ, we get

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma (\mu S - T)^{-1} S\, d\mu &= \frac{1}{2\pi i}\int_{\eta(\Gamma)} z[-I + z(zI - B)^{-1}]\,\frac{dz}{z^2} \\ &= \frac{1}{2\pi i}\int_{\eta(\Gamma)} [-z^{-1} I + (zI - B)^{-1}]\, dz \\ &= -\frac{1}{2\pi i}\int_{\eta(\Gamma)} \frac{1}{z}\, dz\; I + \frac{1}{2\pi i}\int_{\eta(\Gamma)} (zI - B)^{-1}\, dz = -P_{\{(\alpha-\lambda)^{-1}\}}, \end{aligned}$$

where $P_{\{(\alpha-\lambda)^{-1}\}}$ is the spectral projection associated with the operator $(\alpha S - T)^{-1}S$ around $(\alpha-\lambda)^{-1}$. Hence, according to the spectral decomposition theory,

$$P\mathcal{X} = P_{\{(\alpha-\lambda)^{-1}\}}\mathcal{X} = \mathrm{Ker}((\alpha S - T)^{-1}S - (\alpha-\lambda)^{-1}I) = \mathrm{Ker}(T - \lambda S).$$

□

Now, we show some results obtained in the qualitative aspect for the generalized spectrum theory.

We denote by $B(0,k) \subset \mathbb{C}$ the ball with center 0 and radius $k>0$.

Theorem 2.5. *Let $T, S \in BL(\mathcal{X})$, then there exists $k>0$ such that $sp(T,S) \subset B(0,k)$ if and only if $0 \notin sp(S)$.*

P r o o f. We assume that $sp(T,S) \subset B(0,k)$, then for $\alpha \in re(T,S)$, we get

$$\lambda S - T = (\alpha S - T)[(\alpha S - T)^{-1}S - (\alpha-\lambda)^{-1}](\lambda - \alpha). \tag{2.5}$$

As $\alpha \in re(T,S)$, we have that

$$\lambda \in sp(T,S) \Longleftrightarrow (\alpha-\lambda)^{-1} \in sp((\alpha S - T)^{-1}S).$$

So, the inclusion $sp(T,S) \subset B(0,k)$ implies the relation $0 \notin sp((\alpha S - T)^{-1}S)$; otherwise, $0 \in sp((\alpha S - T)^{-1}S)$ implies $\infty \in sp(T,S)$. Thus $0 \notin sp((\alpha S - T)^{-1}S)$ gives $0 \notin sp(S)$.

□

We denote by $sp_p(T,S)$ the set of all generalized eigenvalues. It is clear that when $\mathcal{X}$ is a finite-dimensional space, the generalized spectrum consists only of the generalized eigenvalues, except $\{\infty\}$.

Theorem 2.6. *Let $T, S \in BL(\mathcal{X})$, if S is compact, then*

$$sp(T,S) = sp_p(T,S) \cup \{\infty\}.$$

P r o o f. We use the expression (2.5). Since the operator $(\alpha S - T)^{-1}S$ is compact, $sp(T,S)$ is a set of isolated points. Let $\lambda \in sp(T,S)$, then there is $\gamma \in sp((\alpha S - T)^{-1}S)$, where $\gamma = (\alpha-\lambda)^{-1}$. Hence there exists $u \in \mathcal{X} \setminus \{0\}$ such that

$$\begin{aligned}(\alpha S - T)^{-1}Su = \beta u \ &\Rightarrow \ (\alpha S - T)^{-1}(\alpha S - \lambda S)u = u \\ &\Rightarrow \ u + (\alpha S - T)^{-1}(T - \lambda S)u = u \Rightarrow Tu = \lambda S u.\end{aligned}$$

□

3. Generalized spectrum approximation

Let $T, S \in BL(\mathcal{X})$, where $re(T,S) \neq \emptyset$, and let $(T_k)_{k\in\mathbb{N}}$ and $(S_k)_{k\in\mathbb{N}}$ be two sequences in $BL(\mathcal{X})$. We will use the following conditions:

(H1) S *is a compact operator in* $BL(\mathcal{X})$,

(H2) $\|(T_k - T)x\| \to 0$, $\|(S_k - S)x\| \to 0$ *for all* $x \in \mathcal{X}$,

(H3) $\|(T_k - T)T\| \to 0$,

(H4) $\|(S_k - S)T\| \to 0$.

In the sequel, we write $\cdot \overset{p}{\to} \cdot$ to express the pointwise convergence, while the norm convergence is denoted by $\cdot \overset{n}{\to} \cdot$.

P r o p o s i t i o n 3.1. *(see [6]) Let* $T, \tilde{T}, S, \tilde{S} \in BL(\mathcal{X})$. *For all* $z \in re(T,S)$, *if* $\|R(z,T,S)\left[(T-\tilde{T}) - z(S-\tilde{S})\right]\| < 1$, *then* $z \in re(\tilde{T},\tilde{S})$, *and the next inequality is satisfied*

$$\|R(z,\tilde{T},\tilde{S})\| \leq \frac{\|R(z,T,S)\|}{1 - \|R(z,T,S)\left[(T-\tilde{T}) + z(S-\tilde{S})\right]\|}.$$

R e m a r k 3.1. According to our assumptions in (H1) – (H4) we can easily conclude that

$$[(T_k - T) - \lambda(S_k - S)]\,(T - zS) \overset{n}{\to} 0,$$

for all $z \in re(T,S)$.

P r o p o s i t i o n 3.2. *Let* A, B *and* C *be three bounded operators such that* $0 \notin sp(B)$ *and* $AB \overset{n}{\to} C$, *then* $B^{-1}A \overset{n}{\to} B^{-1}CB^{-1}$.

P r o o f. We note that $||B^{-1}A - B^{-1}CB^{-1}|| \leq ||B^{-1}||\;||AB - C||\;||B^{-1}||$. □

Theorem 3.7. Property U. *For* $k \in \mathbb{N}$, *under* (H1) – (H4), *if* $\lambda_k \in sp(T_k, S_k)$ *and* $\lambda_k \to \lambda$, *then* $\lambda \in sp(T,S)$.

P r o o f. We assume that $\lambda \in re(T,S)$. Since the set $re(T,S)$ is open in $\mathbb{C}$, as stated in Corollary 2.1, there exists $r > 0$ such that

$$E := \{\mu \in \mathbb{C} : |\mu - \lambda| < r\} \subset \mathrm{re}(T,S).$$

On the other side, for all $z \in E$ and for all $k \in \mathbb{N}$, we find that

$$T_k - zS_k = (T - zS)\,[I + R(z,T,S)[(T - T_k) - z(S - S_k)]]\,.$$

Using Remark 3.1 and Proposition 3.2 with

$$A = [(T - T_k) - \lambda(S - S_k)], \quad B = (T - \lambda S),$$

so, there exists $k_0 \in \mathbb{N}$ such that

$$\|R(z,T,S)\,[(T - T_k) - \lambda(S - S_k)]\,\| \leq \frac{1}{2},$$

for all $k \geq k_0$. Then, by Proposition 3.1, we find $z \in re(T_k, S_k)$ such that

$$\|R(z, T_k, S_k)\| \leq 2\|R(z, T, S)\|, \quad \forall k \geq k_0,$$

but $\lambda_k \to \lambda$, thus there exists $k_1 \in \mathbb{N}$ such that $\lambda_k \in E \subset re(T_k, S_k)$ for $k \geq k_1$, which form a contradiction. □

In numerical test, we calculate the quantity

$$\sup \big\{dist(\mu, sp(T, S)) : \mu \in sp(T_k, S_k)\big\},$$

its convergence to 0 implies the Property U. We mention that

$$dist(\mu, sp(T, S)) = \inf_{y \in sp(T,S)} |\mu - y|.$$

Lemma 3.1. *(see [10]) Let P_1 and P_2 be two projections on $\mathcal{X}$ such that*

$$||(P_1 - P_2)P_1|| < 1,$$

then $\dim P_1\mathcal{X} \leq \dim P_2\mathcal{X}$.

Lemma 3.2. *Let $z \in re(T, S)$, under* (H1) – (H4) *there exists a positive integer $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, $z \in re(T_k, S_k)$ and*

$$R(z, T_k, S_k) \xrightarrow{n} R(z, T, S).$$

P r o o f. Let $z \in re(T, S)$, we have

$$T_k - zS_k = (T - zS)\left[I + R(z, T, S)[(T - T_k) - z(S - S_k)]\right],$$

for all $k \in \mathbb{N}$. As stated above in the demonstration of Theorem 3.7, we find $z \in \mathrm{re}(T_k, S_k)$ for all $k \geq k_0$, and $R(z, T_k, S_k)$ is uniformly bounded for all $k \in \mathbb{N}$.

On the other side, for $z \in re(T, S) \cap re(T_k, S_k)$,

$$R(z, T_k, S_k) - R(z, T, S) = R(z, T, S)\left[(T - T_k) - z(S - S_k)\right] R(z, T_k, S_k).$$

Since $R(z, T, S)\left[(T - T_k) - z(S - S_k)\right] \xrightarrow{n} 0$ (according Remark 3.1 and to Proposition 3.2) and $R(z, T_k, S_k)$ is uniformly bounded for all $k \in \mathbb{N}$, we have that

$$R(z, T, S)\left[(T - T_k) - z(S - S_k)\right](T_k - zS_k)^{-1} \xrightarrow{n} 0.$$

□

Theorem 3.8. *Let* λ *be a generalized eigenvalue of finite type, isolated in* $sp(T,S)$. *We denote by* Γ *the Cauchy contour separating* λ *from* $sp(T,S)$. *Under* (H1) – (H4), *there exists* $k_0 \in \mathbb{N}$ *such that for each* $k \geq k_0$, *we have*

$$\dim P\mathcal{X} = \dim P_k\mathcal{X},$$

where

$$P = -\frac{1}{2i\pi}\int_\Gamma R(z,T,S)S\,dz, \quad P_k = -\frac{1}{2\pi i}\int_\Gamma R(z,T_k,S_k)^{-1}S_k\,dz.$$

P r o o f. For $z \in \Gamma$ and $k \geq k_0$, we see that

$$\begin{aligned} R(z,T_k,S_k)S_k - R(z,T,S)S &= [R(z,T_k,S_k) - R(z,T,S)]\,S \\ &- [R(z,T_k,S_k) - R(z,T,S)]\,(S - S_k) - R(z,T,S)(S - S_k). \end{aligned}$$

From (H1) – (H4) we easily find that $(S - S_k)(T - zS) \xrightarrow{n} 0$, thus according to Proposition 3.2 we have $R(z,T,S)(S - S_k) \xrightarrow{n} 0$. Now by using Lemma 3.2, we have

$$R(z,T_k,S_k)S_k - R(z,T,S)S \xrightarrow{n} 0.$$

Finally, we apply Lemma 3.1 and find that $\dim P\mathcal{X} = \dim P_k\mathcal{X}$ for $k \geq k_0$. □

Theorem 3.9. Property L. *Let* λ *be a generalized eigenvalue of finite type, isolated in* $sp(T,S)$. *Under* (H1) – (H4) *there exists a sequence* $\lambda_k \in sp(T_k,S_k)$ *such that* $\lambda_k \to \lambda$.

P r o o f. Let Γ be the Cauchy contour separating λ from $sp(T,S)$. We set

$$\lambda_k \in \mathrm{int}(\Gamma) \cap sp(T_k,S_k).$$

Since $re(T,S) \ni z \mapsto R(z,T,S)S$ and $re(T_k,S_k) \ni z \mapsto R(z,T_k,S_k)S_k$ are analytic functions, and $P_k \xrightarrow{n} P$, we find

$$(\lambda_k)_{k\in\mathbb{N}} = \emptyset \Longleftrightarrow \mathrm{int}(\Gamma) \cap sp(T,S) = \emptyset.$$

We fix $\epsilon > 0$ such that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ belongs to B, where

$$B = \{z \in \mathbb{C} : \, |z - \lambda| \leq \epsilon\}.$$

On the other hand, it is enough to show that every convergent subsequence of $(\lambda_k)_{k\in\mathbb{N}}$ converges to λ itself. Indeed, let a subsequence $(\lambda_{k'})_{k'\in\mathbb{N}}$ converge to $\tilde{\lambda}$ where $\tilde{\lambda} \neq \lambda$. By Property U proved in Theorem 3.7, we see that $\tilde{\lambda} \in sp(T,S)$, but $\tilde{\lambda} \in B$ and $sp(T,S) \cap B = \{\lambda\}$, hence $\lambda = \tilde{\lambda}$, thus $\lambda_k \to \lambda$. □

The last theorem shows that for every generalized eigenvalue λ of finite type isolated in $sp(T,S)$, there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ converging to λ such that $\lambda_k \in sp(T_k,S_k)$. The next result shows that the generalized eigenvectors associated to λ_k converge to the generalized eigenvector associated with λ.

We define the notion of gap between two closed subspaces Z and Y of $\mathcal{X}$ as

$$\mathrm{gap}(Z,Y) = \max\left\{\gamma(Z,Y),\gamma(Y,Z)\right\},$$

where

$$\gamma(Z,Y) = \sup\left\{dist(z,Y):\ z\in Z,\ \|z\|=1\right\}.$$

Theorem 3.10. *Let $M = P\mathcal{X}$ and $M_k = P_k\mathcal{X}$ for $k\in\mathbb{N}$. Then $gap(M,M_k)\to 0$.*

P r o o f. Let $u\in M = P\mathcal{X}$ such that $\|u\|=1$. For $k\in\mathbb{N}$ large enough we have

$$dist(u,M_k) \le \|u-P_ku\| = \|Pu-P_ku\| \le \|P-P_k\|.$$

Let $u\in M_k = P_k\mathcal{X}$ such that $\|u\|=1$. For $k\in\mathbb{N}$ large enough

$$dist(u,M) \le \|u-Pu\| = \|P_ku-Pu\| \le \|P-P_k\|,$$

which implies $\mathrm{gap}(M,M_k)\le \|P_k-P\|$. □

4. Numerical application

As an example for which the numerical results are available by other approaches, we consider the following problem from [11]; it is also studied in [13].

We consider the unbounded operator A defined on $L^2(0,+\infty)$ by the differential equation

$$Au := -u'' + x^2u, \quad x\in[0,+\infty), \quad u(0)=0.$$

This is the harmonic oscillator problem with domain

$$D(A) = H^2(0,+\infty)\cap\left\{u\in L^2(0,\infty):\ \int_0^\infty x^2\,|u|^2dx<\infty\right\}.$$

First, according to the theory of pseudo spectrum for self-adjoint operators (see [6], [11] and [14]) we can find

$$sp(A) = \bigcup_{a>0} sp(A_a), \tag{4.6}$$

where A_a is the Schrödinger operator which has the same formula as A in $L^2(0,a)$, but with the Dirichlet condition at the point a. The domain of A_a is given by

$$D(A_a) = H^2(0,a)\cap H_0^1(0,a).$$

Let $a>0$, we denote by L_a the Laplacien operator defined on $L^2(0,a)$ by

$$L_au = -u'', \quad D(L) = H^2(0,a)\cap H_0^1(0,a).$$

P r o p o s i t i o n 4.3. *(see [12]) L_a is invertible and its inverse is the bounded operator S_a defined by*

$$S_a u(x) = \int_0^a G_{\{0,a\}}(x,y)u(y)dy, \quad u \in L^2(0,a),$$

where

$$G_{\{0,a\}}(x,y) = \begin{cases} \frac{x(a-y)}{a} & 0 \le x \le y \le a, \\ \frac{y(a-x)}{a} & 0 \le y \le x \le a. \end{cases}$$

Let T_a be a bounded operator defined on $L^2(0,a)$ to itself by

$$T_a u(x) = u(x) + \int_0^a G_{\{0,a\}}(x,y)y^2 u(y)dy.$$

Theorem 4.11. $sp(A) = \bigcup_{a>0} sp(T_a, S_a)$.

P r o o f. According to (4.6), we need only to show that $sp(A_a) = sp(T_a, S_a)$ for all $a > 0$.

Let λ be an eigenvalue of A_a with the eigenvector $u \in D(A_a) \setminus \{0\}$. By applying S_a to $A_a u = \lambda u$, we get $T_a u = \lambda S_a u$, which implies that λ is a generalized eigenvalue of the couple (T_a, S_a) with the eigenvector $u \in L^2(0,a) \setminus \{0\}$.

Inversely, let λ be a generalized eigenvalue of the couple (T_a, S_a) with the eigenvector $u \in L^2(0,a) \setminus \{0\}$, i. e. $T_a u = \lambda S_a u$, so

$$u = \lambda S_a u - S_a(v\,u) \;\Rightarrow\; u = S_a(\lambda u - v\,u),$$

where $v(x) = x^2$. Since $\lambda u + vu \in L^2(0,a)$, we have $u \in D(L_a) = D(A_a)$, then

$$u + S_a(v\,u) = \lambda S_a u \;\Rightarrow\; L_a u + v\,u = \lambda u.$$

□

Now, for $a > 0$ we use Kantorovich's projection method to approach the operators T_a and S_a. We define a subdivision of $[0,a]$ for $n \ge 2$ by

$$h_n = \frac{a}{n-1}, \quad x_i = (i-1)h_n, \quad 1 \le i \le n.$$

Let $T_{a,n}$ and $S_{a,n}$ be the approximation operators of T_a and S_a by means of Kantorovich's projection methods (see [9]), given for all $x \in [0,a]$ by

$$\begin{aligned} T_{a,n}u_n(x) &\approx u_n(x) + \sum_{i=1}^n \Big(\int_0^a G_{\{0,a\}}(x_i,y)y^2 u_n(y)dy\Big)e_i(x), \\ S_{a,n}u_n(x) &\approx \sum_{i=1}^n \Big(\int_0^a G_{\{0,a\}}(x_i,y)u_n(y)dy\Big)e_i(x), \end{aligned}$$

where, for $2 \le i \le n-1$,

$$e_i(x) = \begin{cases} 1 - \dfrac{|x-x_i|}{h_n}, & x_{i-1} \le x \le x_{i+1} \\ 0, & otherwise, \end{cases}$$

$$e_1(x) = \begin{cases} \dfrac{x_2 - x}{h_n}, & x_1 \le x \le x_2 \\ 0, & otherwise, \end{cases}$$

$$e_n(x) = \begin{cases} \dfrac{x - x_{n-1}}{h_n}, & x_{n-1} \le x \le x_n \\ 0, & otherwise. \end{cases}$$

By applying Kantorovich's projection method [9] to the equation $T_a u = \lambda S_a u$, we get the approximate equation

$$u_n(x) + \sum_{i=1}^{n} \Big(\int_0^a G_{\{0,a\}}(x_i, y) y^2 u_n(y) dy \Big) e_i(x) = \lambda_n \sum_{i=1}^{n} \Big(\int_0^a G_{\{0,a\}}(x_i, y) u_n(y) dy \Big) e_i(x), \quad x \in [0,a].$$

Denote by β_1 and β_2 the two vectors

$$\beta_1(i) = \int_0^a G_{\{0,a\}}(x_j, y) y^2 u_n(y) dy, \quad \beta_2(i) = \int_0^a G_{\{0,a\}}(x_j, y) u_n(y) dy, \quad 1 \le i \le n,$$

then we can rewrite the previous approximate equation as

$$u_n(x) + \sum_{i=1}^{n} \beta_1(i) e_i(x) = \lambda_n \sum_{i=1}^{n} \beta_2(i) e_i(x). \tag{4.7}$$

Multiplying first equation (4.7) by $G_{\{0,a\}}(x_j, x) x^2$ for $1 \le j \le n$ and integrating over $[0,a]$, we obtain

$$\lambda_n \sum_{i=1}^{n} \beta_2(i) \Big(\int_0^a G_{\{0,a\}}(x_j, x) x^2 e_i(x) dx \Big) = \int_0^a G_{\{0,a\}}(x_j, x) x^2 u_n(x) dx + \sum_{i=1}^{n} \beta_1(i) \Big(\int_0^a G_{\{0,a\}}(x_j, x) x^2 e_i(x) dx \Big).$$

The latter equation is equivalent to the matrix equation

$$\beta_1 + A\beta_1 = \lambda_n A \beta_2, \tag{4.8}$$

where A is a matrix defined by

$$A(i,j) = \int_0^a G_{\{0,a\}}(x_j, x) x^2 e_i(x) dx, \quad 1 \le i, j \le n.$$

In the same way, multiplying equation (4.7) by $G_{\{0,a\}}(x_j, x)$ for $1 \leq j \leq n$ and integrating over $[0, a]$, we also obtain

$$\begin{aligned}\lambda_n \sum_{i=1}^{n} \beta_2(i)\Big(\int_0^a G_{\{0,a\}}(x_j, x)e_i(x)dx\Big) &= \int_0^a G_{\{0,a\}}(x_j, x)u_n(x)dx \\ &+ \sum_{i=1}^{n} \beta_1(i)\Big(\int_0^a G_{\{0,a\}}(x_j, x)e_i(x)dx\Big),\end{aligned}$$

the latter equation is equivalent to the matrix equation

$$\beta_2 + B\beta_1 = \lambda_n B\beta_2, \tag{4.9}$$

where B is a matrix defined by

$$B(i, j) = \int_0^a G_{\{0,a\}}(x_j, x)e_i(x)dx, \quad 1 \leq i, j \leq n.$$

So, by using this process, we have transformed the equation (4.7) into the system of two matrix equations (4.8) and (4.9), namely

$$\begin{cases}\beta_1 + A\beta_1 = \lambda_n A\beta_2, \\ \beta_2 + B\beta_1 = \lambda_n B\beta_2.\end{cases}$$

We also can write this system as

$$\begin{cases}(I_{n\times n} + A)\beta_1 \; + \; O_{n\times n}\beta_2 \; = \lambda_n O_{n\times n}\beta_1 + \lambda_n A\beta_2, \\ \qquad B\beta_1 \qquad + \qquad \beta_2 \qquad = \lambda_n O_{n\times n}\beta_1 + \lambda_n B\beta_2,\end{cases}$$

where $I_{n\times n}$ is the identity matrix with dimension $n \times n$ and $O_{n\times n}$ is the null matrix with dimension $n \times n$. This leads to the matrix generalized eigenvalue problem

$$\begin{bmatrix} A + I_{n\times n} & O_{n\times n} \\ B & I_{n\times n} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \lambda_n \begin{bmatrix} O_{n\times n} & A \\ O_{n\times n} & B \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

Finally, we use the command "eig" in Matlab to calculate the generalized eigenvalue of

$$\left(\begin{bmatrix} A + I_{n\times n} & O_{n\times n} \\ B & I_{n\times n} \end{bmatrix}, \begin{bmatrix} O_{n\times n} & A \\ O_{n\times n} & B \end{bmatrix} \right).$$

We mention that Kantorovich's projection method gives the norm convergence (see [9]) which satisfies our assumption in (H1) − (H4).

We fix $n = 200$ to approach the eigenvalues in our example.

The following table 1 shows that the Kantorovich's method converges perfectly compared with the exact eigenvalue.

Table 1: The numerical results for a=5

Exact eigenvalue	Kantorovich's method
3	3.0001972
7	7.0009887
11	11.0026039
15	15.0103317
19	19.0806050

5. Conclusion

Our study shows the efficiency of the generalized spectrum method, theoretically and numerically. This technique appears to be a computationally attractive tool for resolving the spectral pollution. We resolved this spectral pollution by treating the analytical question: to find the bounded operators T and S representing the spectrum proprieties of an unbounded operator A in the theory of generalized spectrum.

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